Constraints on possible dynamics of QCD by symmetry and anomaly

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Message

Quantum field theorists are interested in low-energy behaviors of collective phenomena:

Problem

Does a QFT have the unique gapped ground state?

Both answers (Yes and No) to this question are equally interesting. On the other hand, I want to convince that

- It is almost always HARD to answer YES.
- It is often EASY to answer NO.
  Give thanks to Anomaly matching!

Outline

- Modern view on anomaly matching
- Examples
  - QM on circle, Single spin system
  - Perturbative chiral anomalies in even dimensions, Original anomaly matching in QCD
- My result on QCD with massless quarks (1807.07666)
  - New discrete 't Hooft anomaly of massless QCD
  - Anomaly matching in ordinary chiral symmetry breaking
  - No-go for Stern phase (exotic chiral-broken phase)

Anomaly matching as a theorem of QFT

't Hooft anomaly

Let us first define the 't Hooft anomaly:

Definition

Symmetry G of QFT has an 't Hooft anomaly
$\Leftrightarrow$ *Partition function $\mathcal{Z}_d[A]$ with the background G-gauge field A is anomalous:*

$$\mathcal{Z}_d[A + \mathrm{d}_A\theta] = \mathcal{Z}_d[A] \exp\left(\mathrm{i}\mathcal{A}_d[\theta, A]\right),$$

with $\mathcal{A}_d[\theta, A] \neq \delta_\theta$(any d-dim. local functional of A).

Comments:

- When we want to dynamically gauge G, the anomaly is called gauge anomaly.

Anomaly matching

Theorem

We consider the RG flow: $QFT_{UV} \to QFT_{IR}$. Then,

$$\mathcal{A}_{d,\mathrm{UV}}[\theta, A] = \mathcal{A}_{d,\mathrm{IR}}[\theta, A].$$

This is called the 't Hooft anomaly matching condition ('t Hooft, '80, ...).

Let us show this statement under the anomaly-inflow assumption (Callan, Harvey, '85), which is empirically always true:

Assumption

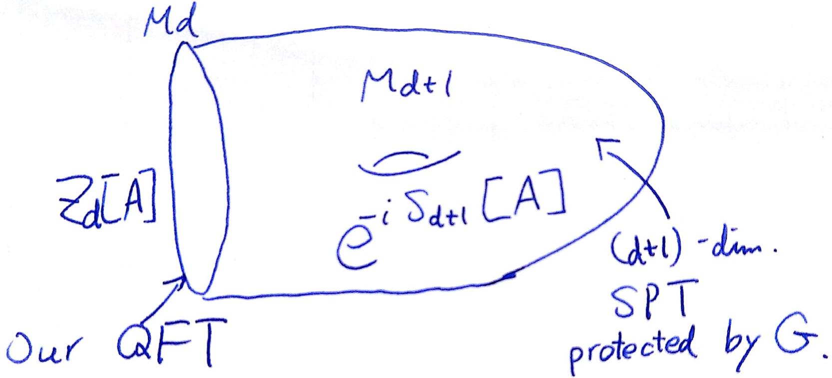
Anomaly $\mathcal{A}_d[\theta, A]$ in d-dim. satisfies

$$\mathcal{A}_d[\theta, A] = \delta_\theta(S_{d+1}[A]),$$

where $S_{d+1}[A]$ defines a $(d+1)$-dim. topological G-gauge theory.

Anomaly inflow

When gauging G, we regards our theory as a boundary of $(d+1)$-dim. symmetry-protected topological (SPT) phase protected by G:



(Wen, '13, Kapustin, Thorngren, '14, Cho, Teo, Ryu, '14, ...)

Proof of anomaly matching

RG flow, $\mathrm{QFT}_{UV} \to \mathrm{QFT}_{IR}$, means that, as we let M_d larger,

$$\mathcal{Z}_{d,\mathrm{UV}}[A]\exp(-\mathrm{i}S_{d+1}[A]) \to \mathcal{Z}_{d,\mathrm{IR}}[A]\exp(-\mathrm{i}S_{d+1}[A]),$$

since the contribution from massive particle excitations drop out.

Anomaly inflow says the UV partition function is gauge-invariant:

$$\begin{aligned} 0 &= \delta_\theta(\ln\{\mathcal{Z}_{d,\mathrm{UV}}[A]\exp(-\mathrm{i}S_{d+1}[A])\}) \\ &\to \delta_\theta(\ln\{\mathcal{Z}_{d,\mathrm{IR}}[A]\exp(-\mathrm{i}S_{d+1}[A])\}) \\ &= \mathrm{i}(\mathcal{A}_{d,\mathrm{IR}} - \mathcal{A}_{d,\mathrm{UV}}). \end{aligned}$$

This gives $\mathcal{A}_{d,\mathrm{UV}} = \mathcal{A}_{d,\mathrm{IR}}$.

Results of anomaly matching

Anomaly matching claims that QFT_{IR} must also have a nontrivial 't Hooft anomaly of symmetry G.

This excludes the unique and gapped ground state.

Possible scenarios for IR physics are

- Gappless,
- Ground states degeneracy by spontaneous breaking,
- Ground states degeneracy by topological order,

or their combinations.

Examples 1. Quantum mechanics on a circle

Quantum mechanics on S^1

We consider quantum mechanics of particle $x \in S^1$, i.e. $x \sim x + 2\pi$.
Lagrangian:

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{\theta}{2\pi}\dot{x}.$$

Canonical momentum is given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x} + \frac{\theta}{2\pi}.$$

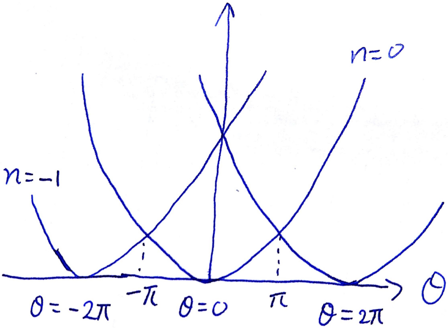
We obtain the Hamiltonian

$$H = \frac{1}{2}\left(p - \frac{\theta}{2\pi}\right)^2.$$

Quantum mechanically, $E_n = \frac{1}{2}(n - \frac{\theta}{2\pi})^2$ with the eigenstate $\mathrm{e}^{\mathrm{i}nx}$.

Degeneracy of states at $\theta = \pi$

Energy spectrum looks as



- At generic θ, the ground states are unique.
- At $\theta = \pi$, the ground states are doubly degenerate.
  ($\Leftarrow$ Anomaly matching (Gaiotto, Kapustin, Komargodski, Seiberg, '17))

Anomaly of $U(1) \rtimes \mathbb{Z}_2$ at $\theta = \pi$

At $\theta = \pi$, the system has

- $U(1)$ symmetry, $x \mapsto x + \alpha$, and
- $\mathbb{Z}_2$ charge conjugation, $x \mapsto -x$.

Let us introduce the $U(1)$ gauge field A by minimal coupling, then

$$S[A] = \int \mathrm{d}\tau \frac{1}{2}(\dot{x} + A_0)^2 - \frac{\mathrm{i}\theta}{2\pi}\int (\mathrm{d}x + A).$$

Charge conjugation C at $\theta = \pi$ gives

$$C : S[A] \to S[A] - \mathrm{i}\int (\mathrm{d}x + A) = S[A] - \mathrm{i}\int A \pmod{2\pi}.$$

This gives the 't Hooft anomaly:

$$C : \mathcal{Z}_{1\mathrm{d\ QM},\theta=\pi}[A] \mapsto \mathcal{Z}_{1\mathrm{d\ QM},\theta=\pi}[A] \mathrm{e}^{-\mathrm{i}\int A}.$$

Anomaly inflow

In order to get C invariance with A, we need to add $+\frac{\mathrm{i}}{2}\int A$ to the Lagrangian, but it breaks $U(1)$ gauge invariance.

In order to establish both invariance, we must consider 2-dim. bulk term,

$$\frac{1}{2}\int_{M_2}\mathrm{d}A,$$

and then the combined system

$$\mathcal{Z}_{\mathrm{1d\ QM}}[A]\exp\left(\frac{\mathrm{i}}{2}\int_{M_2}\mathrm{d}A\right)$$

is both $U(1)$ gauge invariant and C-invariant.

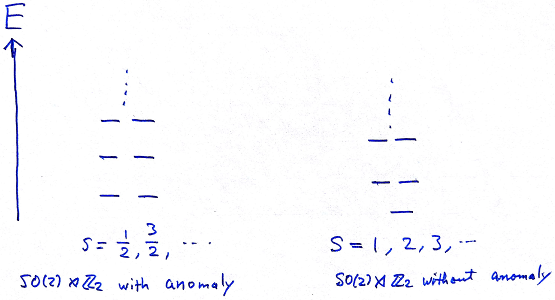
- Anomaly can show that the degeneracy at $\theta=\pi$ is persistent under any local perturbations preserving $\mathbb{Z}_2 \rtimes \mathbb{Z}_2 \subset U(1) \rtimes \mathbb{Z}_2$.

Single spin system: Half-integer and integer spin

Consider single spin with $H = J\hat{S}_z^2$.

Spin symmetry is $SO(3)$ when $J=0$ (not $SU(2)$ even for $S=1/2$), and $J>0$ breaks it as $SO(2)\rtimes \mathbb{Z}_2 \subset SO(3)$:

$$\mathsf{C} : \mathcal{Z}_{\mathrm{spin}\,S}[A] \mapsto \mathcal{Z}_{\mathrm{spin}\,S}[A] \exp\left(-\mathrm{i}(2S)\int A\right).$$



Examples 2. (Perturbative) Chiral anomaly in even dimensions

(Perturbative) chiral anomaly

Let's consider left-handed fermions in $2n$ dimension:

$$\overline{\psi}\gamma_\mu\partial_\mu P_\mathrm{L}\psi.$$

If it has $SU(N)$ symmetry, then we consider the minimal coupling to the $SU(N)$ gauge field A,

$$\overline{\psi}\gamma_\mu(\partial_\mu + A_\mu)P_\mathrm{L}\psi,$$

and define the partition function $\mathcal{Z}_{2n}[A]$. Chiral anomaly is defined by its infinitesimal gauge transformation as

$$\mathcal{A}_{2n}[\theta, A] = -\mathrm{i}\delta_\theta \ln \mathcal{Z}_{2n}[A].$$

- Regarding θ as a FP ghost, δ_θ is BRST transformation: $\delta_\theta^2 = 0$.
- Consistent anomaly: $\delta_\theta\mathcal{A}_{2n} = -\mathrm{i}\delta_\theta^2 \ln \mathcal{Z}_{2n} = 0$ (Wess, Zumino, '71).

Stora-Zumino chain and Anomaly inflow

Consistency condition can be solved in *ad hoc* way by Stora-Zumino chain (Stora, '83, Zumino, '83, Alvarez-Gaume, Ginsparg, '85): $\mathcal{A}_{2n} = \int \Omega^1_{2n}(\theta, A)$.

$$\begin{tikzcd}
\delta_\theta \Omega^1_{2n} \arrow[r, "\mathrm{d}"] & 0 & & \\
\color{blue}{\Omega^1_{2n}} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \delta_\theta \mathrm{CS} \arrow[u, "\delta"] \arrow[r, "d"] & 0 & \\
& \mathrm{CS}_{2n+1} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \mathrm{tr}[F^{n+1}] \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & 0
\end{tikzcd}$$

Non-Abelian chiral anomaly thus satisfies anomaly-inflow assumption:

$$\mathcal{Z}_{2n}[A] \exp\left(-\mathrm{i} \int_{M_{2n+1}} \mathrm{CS}_{2n+1}[A]\right)$$

is gauge invariant.

Anomaly matching in QCD

QCD with N_f massless quarks has $SU(N_\mathrm{f})_\mathrm{L}\times SU(N_\mathrm{f})_\mathrm{R}$ chiral symmetry with 't Hooft anomaly:

$$\mathcal{Z}_{\mathrm{4d\ QCD}}[A_\mathrm{L},A_\mathrm{R}]\exp\left(\mathrm{i}N_\mathrm{c}\left\{\mathrm{CS}_5[A_\mathrm{R}]-\mathrm{CS}_5[A_\mathrm{L}]\right\}\right).$$

If the low-energy excitations are given by color-singlet particles, they must contain ('t Hooft, '80, Frishman, Schwimmer, Banks, Yankielowicz, '81, ...)

- massless fermions, or
- Nambu-Goldstone bosons.

Remarks:

- NG bosons can always match the anomaly by Wess-Zumino term. (Wess, Zumino, '71)
- For $\mathcal{N}=1$ SQCD with $N_\mathrm{f}=N_\mathrm{c}+1$, the s-confining phase realize the first case. (Seiberg, '94)

Application Discrete anomaly of QCD and no-go for Stern phase

(YT, 1807.07666)

Chiral symmetry breaking of QCD

In the massless quark limit, QCD has chiral symmetry,

$$G = \frac{SU(N_\mathrm{f})_\mathrm{L} \times SU(N_\mathrm{f})_\mathrm{R} \times U(1)_\mathrm{V}}{\mathbb{Z}_{N_\mathrm{c}} \times \mathbb{Z}_{N_\mathrm{f}}}.$$

If N_f is not so large, quark bilinear condensate appears,

$$\langle \overline{q_\mathrm{R}} q_\mathrm{L} \rangle = -\Lambda^3,$$

and it breaks this symmetry spontaneously,

$$G \to H = \frac{SU(N_\mathrm{f})_\mathrm{V} \times U(1)_\mathrm{V}}{\mathbb{Z}_{N_\mathrm{c}} \times \mathbb{Z}_{N_\mathrm{f}}}.$$

Exotic chiral symmetry breaking

Theoretically, it is also possible to consider very similar but different pattern of chiral symmetry breaking (Stern, 97, 98):

- Order parameter is quark quartic:

$$\sum_a (\overline{q_\mathrm{R}} T^a q_\mathrm{L})(\overline{q_\mathrm{L}} T^a q_\mathrm{R}).$$

  The symmetry breaking pattern is

$$G \to G^\mathrm{sub} \equiv \underbrace{\frac{SU(N_\mathrm{f})_\mathrm{V} \times U(1)_\mathrm{V}}{\mathbb{Z}_{N_\mathrm{c}} \times \mathbb{Z}_{N_\mathrm{f}}}}_{H} \times (\mathbb{Z}_{N_\mathrm{f}})_L.$$

- This is ruled out at the zero-density QCD, i.e., when QCD inequality is valid. (Kogan, Kovner, Shifman, 98)

Purpose

We rule out the Stern phase based only on symmetry and anomaly.

Symmetry of massless QCD

QCD Lagrangian:

$$S = \frac{1}{2g^2} \int \mathrm{tr}(G \wedge *G) + \int \sum_{f=1}^{N_\mathrm{f}} \overline{q}_f \gamma_\mu D_\mu q.$$

Symmetry of massless QCD (i.e. Lagrangian + $\mathcal{D}\overline{q}\mathcal{D}q$):

$$G = \frac{SU(N_\mathrm{f})_\mathrm{L} \times SU(N_\mathrm{f})_\mathrm{R} \times U(1)_\mathrm{V}}{\mathbb{Z}_{N_\mathrm{c}} \times (\mathbb{Z}_{N_\mathrm{f}})_\mathrm{V}}$$

- Due to the gauge invariance, symmetry must be divided by $\mathbb{Z}_{N_\mathrm{c}} \subset SU(N_\mathrm{c}) \times U(1)_\mathrm{V}$.
- In other words, correct $U(1)$ symmetry is the baryon number symmetry $U(1)_B = U(1)_\mathrm{V}/\mathbb{Z}_{N_\mathrm{c}}$ but not $U(1)_\mathrm{V}$.

Contents of background gauge fields

To emphasize the role of discrete axial symmetry, we look at

$$G^{\mathrm{sub}} = \frac{SU(N_{\mathrm{f}})_V \times U(1)_V}{(\mathbb{Z}_{N_{\mathrm{c}}}) \times (\mathbb{Z}_{N_{\mathrm{f}}})} \times (\mathbb{Z}_{N_{\mathrm{f}}})_L.$$

To detect the 't Hooft anomaly of G^{sub}, we introduce the background gauge fields (1710.08923, 1711.10487) (ref. Kapustin, Seiberg (2014)).

Vector part:

- $SU(N_{\mathrm{f}})_V$ one-form gauge field: A_{f}
- $U(1)_V$ one-form gauge field: A_V
- $(\mathbb{Z}_{N_{\mathrm{c}}})$ two-form gauge field: B_{c}
- $(\mathbb{Z}_{N_{\mathrm{f}}})$ two-form gauge field: B_{f}

Chiral part:

- $(\mathbb{Z}_{N_{\mathrm{f}}})_L$ one-form symmetry: A_χ

Two-form gauge fields: Transition functions

Question What is the meaning of two-form gauge fields?

Connection formula on double overlaps of patches $U_i \cap U_j$ $(g_{ij}^{(c)} \in SU(N_c), g_{ij}^{(f)} \in SU(N_f), g_{ij}^{(V)} \in U(1)_V)$:
For gauge field,

$$\begin{aligned} a^{(j)} &= g_{ij}^{(c)\,-1} a^{(i)} g_{ij}^{(c)} + g_{ij}^{(c)\,-1} \mathrm{d} g_{ij}^{(c)}, \\ A_f^{(j)} &= g_{ij}^{(f)\,-1} A_f^{(i)} g_{ij}^{(f)} + g_{ij}^{(f)\,-1} \mathrm{d} g_{ij}^{(f)}, \\ A_V^{(j)} &= g_{ij}^{(V)\,-1} A_V^{(i)} g_{ij}^{(V)} + g_{ij}^{(V)\,-1} \mathrm{d} g_{ij}^{(V)}. \end{aligned}$$

For quark field,

$$q^{(j)} = \left(g_{ij}^{(c)} \otimes g_{ij}^{(f)} \otimes g_{ij}^{(V)} \right)^{-1} q^{(i)}.$$

Two-form gauge fields: Cocycle condition

Consistency requires the cocycle condition for $\left(g_{ij}^{(c)}\otimes g_{ij}^{(f)}\otimes g_{ij}^{(q)}\right)$, but not for each of them:

$$g_{ij}^{(c)}g_{jk}^{(c)}g_{ki}^{(c)}=\exp\left(\frac{2\pi\mathrm{i}}{N_C}n_{ijk}^{(c)}\right)\in\mathbb{Z}_{N_C},$$
$$g_{ij}^{(f)}g_{jk}^{(f)}g_{ki}^{(f)}=\exp\left(\frac{2\pi\mathrm{i}}{N_F}n_{ijk}^{(f)}\right)\in\mathbb{Z}_{N_F},$$
$$g_{ij}^{(V)}g_{jk}^{(V)}g_{ki}^{(V)}=\exp\left(-\frac{2\pi\mathrm{i}}{N_C}n_{ijk}^{(c)}-\frac{2\pi\mathrm{i}}{N_F}n_{ijk}^{(f)}\right)\in U(1)_V.$$

$[\{(n_{ijk}^{(c)},U_i\cap U_j\cap U_k)\}_{ijk}]$ and $[\{(n_{ijk}^{(f)},U_i\cap U_j\cap U_k)\}_{ijk}]$ are characterized by

$$B_c\in H^2(M_4,\mathbb{Z}_{N_C}),\ B_f\in H^2(M_4,\mathbb{Z}_{N_F}).$$

Discrete 't Hooft anomaly for massless QCD

After introducing the background gauge fields, the partition function $\mathcal{Z}_{\mathrm{QCD}}$ is no longer gauge invariant.

Gauge invariance requires to add 5d SPT phase:

$$\mathcal{Z}_{\mathrm{QCD}} \exp\left(\frac{N_{\mathrm{f}}}{(2\pi)^2}\int A_\chi \wedge \underbrace{N_{\mathrm{c}}(\mathrm{d}A_{\mathrm{V}}+B_{\mathrm{c}})}_{\mathrm{d}A_B} \wedge B_{\mathrm{f}}\right).$$

This says that the baryon number conservation is anomalously violated under $(\mathbb{Z}_{N_{\mathrm{f}}})_L$ and $SU(N_{\mathrm{f}})/\mathbb{Z}_{N_{\mathrm{f}}}$ gauge field:

$$\partial_\mu J^\mu_B = \frac{N_{\mathrm{f}}}{(2\pi)^2}\mathrm{d}A_\chi \wedge B_{\mathrm{f}}.$$

(For different but related anomalies, see also Gaiotto, Kapustin, Komargodski, Seiberg '17, Tanizaki, Kikuchi, '17, Shimizu, Yonekura, '17, Komargodski, Sulejmanpasic, Unsal, '17, ...)

Ordinary chiral symmetry breaking and Skyrmions

Let us see how the discrete anomaly is matched for ordinary case:

$$G=\frac{SU(N_\mathrm{f})_L\times SU(N_\mathrm{f})_R\times U(1)_V}{\mathbb{Z}_{N_\mathrm{c}}\times\mathbb{Z}_{N_\mathrm{f}}}\to H=\frac{SU(N_\mathrm{f})_V\times U(1)_V}{\mathbb{Z}_{N_\mathrm{c}}\times\mathbb{Z}_{N_\mathrm{f}}}$$

The nonlinear Langrangian has the target space $G/H=SU(N_\mathrm{f})$.

Since $\pi_3(G/H)=\mathbb{Z}$, there are Skyrmions with the skyrmion current

$$J_\mathrm{skyrmion}=\frac{1}{24\pi^2}\mathrm{tr}[(U^\dagger \mathrm{d}U)^3].$$

We identify Skyrmions as baryons in the chiral Lagrangian description (Skyrme, 61, 62, Witten, '83).

Anomaly matching and skyrmion current

We define the gauge-invariant Skyrmion current with the background gauge field as

$$\begin{aligned} J_{\mathrm{skyrmion}} &= \frac{1}{24\pi^2}\mathrm{tr}[(U^{-1}DU)^3] \\ &+\frac{1}{8\pi^2}\mathrm{tr}[(UDU^{-1})(F_{\mathrm{f}}+\mathrm{d}A_\chi)-(U^{-1}DU)F_{\mathrm{f}}]. \end{aligned}$$

As a result,

$$\partial_\mu J^\mu_{\mathrm{skyrmion}} = \frac{N_{\mathrm{f}}}{(2\pi)^2}\mathrm{d}A_\chi \wedge B_{\mathrm{f}}.$$

$\Rightarrow \mathrm{d}J_{\mathrm{skyrmion}}[A_\chi, B_{\mathrm{f}}] = \mathrm{d}J_B[A_\chi, B_{\mathrm{f}}]$, and the anomaly matching is satisfied. (1807.07666[hep-th])

This is an extension of the $U(1)_V$-$SU(N_{\mathrm{f}})_L$-$SU(N_{\mathrm{f}})_L$ anomaly matching in the chiral Lagrangian. (ref. Witten, 1983)

Exotic chiral symmetry breaking

We now consider the Stern phase, where

$$G \to G^{\mathrm{sub}} = \frac{SU(N_F)_V \times U(1)_V}{(\mathbb{Z}_{N_\mathrm{c}}) \times (\mathbb{Z}_{N_\mathrm{f}})} \times (\mathbb{Z}_{N_\mathrm{f}})_L.$$

The target space of the nonlinear Lagrangian is

$$G/G^{\mathrm{sub}} = SU(N_\mathrm{f})/\mathbb{Z}_{N_\mathrm{f}}.$$

$\Rightarrow$ We can obtain the effective theory by gauging $\mathbb{Z}_{N_\mathrm{f}}$ in the ordinary chiral Lagrangian.

U : Nonlinear sigma field $\in SU(N_\mathrm{f})$,

a_χ : $\mathbb{Z}_{N_\mathrm{f}}$ dynamical gauge field

Mismatch of anomaly in Stern phase

Nontrivial homotopy: $\pi_3(G/G^{\mathrm{sub}})=\mathbb{Z}$, $\pi_1(G/G^{\mathrm{sub}})=\mathbb{Z}_{N_\mathrm{f}}$.
There are skyrmions with the current J_{skyrmion}.
There are also $\mathbb{Z}_{N_\mathrm{f}}$ domain walls.

Under the background gauge fields, the anomalous violation of J_{skyrmion} is

$$\mathrm{d}J_{\mathrm{skyrmion}}=\frac{N_\mathrm{f}}{(2\pi)^2}\mathrm{d}a_\chi\wedge B_\mathrm{f}\neq \mathrm{d}J_B.$$

Therefore, the anomaly matching is not satisfied.

$\Rightarrow$ We rule out the Stern phase from the possible QCD vacua.
This proof is valid even at finite densities. (1807.07666[hep-th])

Summary

- Anomaly matching shows non-existence of unique gapped ground state.

- QCD has a new discrete 't Hooft anomaly, and it can be matched in ordinary chiral-broken phase:

$$\mathrm{d}J_{\text{skyrmion}} = \mathrm{d}J_B = \frac{N_{\mathrm{f}}}{(2\pi)^2}\mathrm{d}A_\chi \wedge B_{\mathrm{f}}.$$

- In the Stern phase, we show that

$$\mathrm{d}J_{\text{skyrmion}} = \frac{N_{\mathrm{f}}}{(2\pi)^2}\mathrm{d}a_\chi \wedge B_{\mathrm{f}} \neq \mathrm{d}J_B.$$

Anomaly matching thus rules it out even at finite densities.